Advanced Logic, Summary

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$$\text{WFFS inductively.}\begin{cases}\text{1.propositional atoms is wff in language...}\\ \text{2.If } P \text{ wff in language..., then so is } \neg P\\ \text{3.If } P, Q \text{ WFFS in language.. then so}\\ \text{4.Nothing is WWF in language.... unless ...}\end{cases}$$

$v(\neg P) = 1 - v(P), v((P \wedge Q)) = \min\{v(P), v(Q)\}, v((P \wedge Q)) = \max\{v(P), v(Q)\}$
$v((P \to Q)) = \max\{1 - v(P), v(Q)\}, v((P \leftrightarrow Q)) = 1 - |v(P) - v(Q)|$

Logic def. by structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\rangle$. $\mathcal{V}$ set of truth values, $\mathcal{D} \subseteq \mathcal{V}$ desig. values, $\mathcal{C}$ set of connectives, $\forall c \in \mathcal{C}$ truth function $f_c : \mathcal{V}^n \to \mathcal{V}$

Example: propositional logic: $\mathcal{V} = \{0,1\}, \mathcal{D} = \{1\}, \mathcal{C} = \{\wedge, \vee, \neg, \supset\}$, so $f_\wedge(x,y) = \min(x,y), f_\vee(x,y) = \max(x,y), f_\neg(x) = 1 - x, f_\supset(x,y) = \max(1-x, y)$

Note:$v(c(A_1, \dots, A_n)) = f_c(v(A_1), \dots, v(A_n))$.

$\Sigma \vDash$ iff (every v s.t. $v(B) \in \mathcal{D}$ for all $B \in \Sigma$, then $v(A) \in \mathcal{D}$)

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	i,0	0
i	1	i,1,i	i,0
0	1	1	1

If the entry in the table does not have the colour corresponding to the name, then you have to take the black value.

Kleene, $\mathcal{V} = \{0, i, 1\}, \mathcal{D} = \{1\}$ and LP, $\mathcal{C} = \{0, i, 1\}, \mathcal{D} = \{i, 1\}$
Lukasiewicz, L_3, $\mathcal{V} = \{0, i, 1\}, \mathcal{D} = \{1\}$
RM3, $\mathcal{V} = \{0, 1, i\}, \mathcal{D} = \{i, 1\}$.

Tableux rules for prop connectives:
Remember that $A \supset B$ implies $\neg A$ or B.
$A \equiv B$ implies both A, B or both $\neg A, \neg B$.
complete: every rule that can be applied has been applied.
Prop. language. Branch is closed, if for some A, both $A, \neg A$ occur on its nodes.

4 valued logic (FDE):
$q\rho 1$: means q related to true, $q\rho 0$, means q related to false.
$(A \wedge B)\rho 1$ iff $A\rho 1 \& B\rho 1$,$(A \wedge B)\rho 0$ iff $A\rho 0$ or,$B\rho 0$.
$(A \vee B)\rho 1$ iff $A\rho 1$ or $B\rho 1$, $(A \vee B)\rho 0$ iff $A\rho \& B\rho 0$.
$(\neg A)\rho 1$ iff $A\rho 0$, and $(\neg A)\rho 0$ iff $A\rho 1$.
Semantic tableaux immediately follows, Morgan's law can be applied.
Branch of tableaux is closed, if both $A, +$ and $A, -$ for some formula A.
To test $A_1, \ldots, A_n \vdash_{\text{FDE}} B$, we start with $A_i, +$ and $B, -$.
Countermodel: For atoms p, if branch contains $p, +$ set $p\rho 1$. If branch contains $\neg p, +$ set $p\rho 0$.
No other facts about ρ obtained.
K3 Closed: Contains both $A, +$ and $\neg A, +$
LP Closed: Contains both $A, +$ and $A, -$
For L3 and RM3 we will get the rules the FDE tabels for $\supset$.

Fuzzy logic: $\mathcal{V} = [0, 1]$, with $f_\neg(x) = 1 - x$, $f_\wedge(x, y) = \min(x, y)$
$f_\vee(x, y) = \max(x, y)$ and $f_\rightarrow(x, y) = \min(1, 1 - x + y)$

$D = [\epsilon, 1]$ then $\Sigma \vDash A$ iff (for all v , if $v(B) \geq \epsilon$ for all ,$B \in \Sigma$, then $v(A) \geq \sigma$)

$\epsilon = 1$, and therefore $\mathcal{D} = \{1\}$ then we have $Ł_\mathcal{N}$

$\Box$ necessity, $\Diamond$,possibility.

Language:

1. Each propositional atom is WFF
2. A is WFF of L, then so are $\neg A, \Box A, \Diamond A$
3. A, B wff of L, then so are $(A \wedge B), (A \vee B), (A \supset B), (A \equiv B)$
4. Nothing is wff of L, unless combination of above.

World W, noenmpty, worlds mentioned. $R \subset W \times W$ and $v : (W \times P) \rightarrow \{0, 1\}$
$v_w(p) = 1$ means p is true at world w.

Extra rules:
$v_w(\Diamond A) = 1$ iff (there is a world $w' \in W$ such that wRw' and $v_{w'}(A) = 1$).
$v_w(\Box A) = 1$ iff (for every world $w' \in W$ such that wRw' it holds that $v_{w'}(A) = 1$.)

$\Sigma \vDash A$ iff
(for all models $\langle W, R, v\rangle$ and all $w \in W$: if $v_w(B) = 1$, for all $B \in \Sigma$, then $v_w(A) = 1$)

Tableau rules same. Extra:

$$\begin{tikzcd}
\neg\Box A, i \arrow[dd] & \neg\Diamond A, i \arrow[dd] & \Box A, i & \Diamond A, i \arrow[d] \\
 & & irj \arrow[d] & irj \\
\Diamond\neg A, i & \Box\neg A, i & A, j & A, j
\end{tikzcd}$$

Branch closed if A, i and $\neg A, i$ both occur on branch for same i.

Countermodel:
$W = \{w_i | i \text{ on branch}\}, R = \{\langle w_i, w_j\rangle | irj \text{ occurs on branch}\}$
For propositional atoms p,so if p, i occurs on branch,then $v_{w_i}(p) = 1$. If $\neg p, i$ occurs on branch, then $v_{w_i}(p) = 0$. Otherwise you can choose $v_{w_i}(p)$ arbitrarily.

ρ means R reflexive, iff for all $w \in W, wRw$.
σ means R symmetric iff for all $w_1, w_2 \in W$ (If w_1Rw_2, then w_2Rw_1)
τ means R transitive, iff for all $w_1, w_2, w_3 \in W$ (if w_1Rw_2, w_2Rw_3, then w_1Rw_3)
η means R extendable, iff for all $w_1 \in W$, there is $w_2 \in W$ s.t. w_1Rw_2.

Note that $\eta, \tau, \sigma \Rightarrow \rho$
υ means R universal, iff w_1Rw_2 for all $w_1, w_2 \in W$.
φ : means R forward convergent iff for all $x, w, y \in W$, if xRy and Rz, then (zRy or $y = z$ or yRz).
β : means R backward convergent iff for all $x, w, y \in W$, if yRx, zRx then (zRy or $y = z$ or yRz).
δ : means R is dense, iff for all $w, z \in W$ (if wRz then there is $y \in W$ s.t. wRy and yRz)

$K_{\rho\sigma\tau}$ is called S5, so reflexive, symmetric and transitive.

Tense logic:
$[..]A$ at all.. times A, $\langle..\rangle A$ at some ... times A.
So if .. = P, then earlier times, .. = F, future times.
$v_w([P]A) = 1$ iff for all w' s.t. $w'Rw, v_{w'}(A) = 1$
$v_w([F]A) = 1$ iff for all w' s.t. $wRw', v_{w'}(A) = 1$
$v_w(\langle P\rangle A) = 1$ iff for some w' s.t. $w'Rw, v_{w'}(A) = 1$
$v_w(\langle F\rangle A) = 1$ iff for some w' s.t. $wRw', v_{w'}(A) = 1$.

$$\begin{tikzcd} {[F]A, i} \\ irj \arrow[d] \\ A, j \end{tikzcd} \qquad \begin{tikzcd} \langle F\rangle A, i \arrow[d] \\ irj \\ A, j \end{tikzcd} \qquad \begin{tikzcd} \neg[F]A, i \arrow[d] \\ \langle F\rangle\neg A, i \end{tikzcd} \qquad \begin{tikzcd} \neg\langle F\rangle A, i \arrow[d] \\ {[F]\neg A, i} \end{tikzcd}$$

If we replace F by P, and replace irj by jri, then we have all rules.
$[..]$,apply to all on branch, $\langle..\rangle$ apply to new on branch.

v arbitrary, b branch of tableau. v faithful to b, iff every formula D, that occurs on $b, v(D) = 1$
If v faithful to b,tableau rule applied to b, then v faithful to at least 1 of generated branches.
b branch v induced by b, if for every p, we have if p on branch, then $v(p) = 1$, if $\neg p$ on branch then $v(p) = 0$. Otherwise arbitrary.
If b complete, result also holds for D, instead of p.
$I = \langle W, R, v\rangle$ and b any branch. Then I faithful to b if there is $f : \mathbb{N}_{\geq 0} \to W$ s.t.

- For every node D, I on b, we have D is true at world $f(i)$ in I
- If irj on b, then $f(i)Rf(j)$ in I

So we say f shows I faithful to b.
I faithful to b, tableau rule applied to b, then at least 1 extension of b, call it b' also satisfy I faithful to b'
$I = \langle W, R, v\rangle$ induced by b iff:

- $W = \{w_i : i \text{ occurs on } b\}$
- $w_R w_j$ iff irj occurs on b.
- p, i on b, then $v_{w_i}(p) = 1$, if $\neg p, i$ on b, then $v_{w_i}(p) = 0$. Other. arbit.

b open complete, $I = \langle W, R, v\rangle$ induced by b. Then for all D, for all i following holds: D, i on b, then $v_{w_i}(D) = 1$, if $\neg D, i$ on b then $v_{w_i}(D) = 0$.
$v_w(Pa_1 \ldots a_n) = 1$ iff $\langle v(a_1), \ldots, v(a_n)\rangle \in v_w(P)$.

$v_w(\exists xA) = 1$ iff for some $d \in D, v_w(A_x(k_d)) = 1$
$v_w(\forall xA) = 1$ iff for all $d \in D, v_w(A_x(k_d)) = 1$.
$A_x(k_d)$ is formula by substituting k_d for each free occurence of x in A, where k_d is constant s.t. $v(k_d) = d$.

First-order version: CK.
$\Sigma \vDash A$ iff for every $I = \langle D, W, R, v\rangle$, and all $w \in W$: if $v_w(B) = 1$ for all $B \in \Sigma$, then $v_w(A) = 1$.

$$\begin{tikzcd}
\neg\exists xA, i \arrow[d] & \neg\forall xA, i \arrow[d] & \forall xA, i \arrow[d] & \exists xA, i \arrow[d] \\
\forall x\neg A, i & \exists x\neg A, i & \begin{array}{c} A_x(a), i \\ \text{old } a \text{ oth. new} \end{array} & \begin{array}{c} A_x(c), i \\ \text{new } c \end{array}
\end{tikzcd}$$

Variable domain version K, called VK.
$I = \langle D, W, R, v\rangle, D \neq 0$ (domain of quant.), $W \neq \varnothing, R \subseteq W \times W$
$v(w) \subseteq D, v(c) \in D, v_w(P) \subseteq D^n, v_w(\mathfrak{E}) = D_w$.

$v_w(\exists xA) = 1$ iff for some $d \in D_w, v_w(A_x(k_d)) = 1$
$v_w(\forall xA) = 1$ iff for all $d \in D_w, v_w(A_x(k_d)) = 1$.
$v_w(\mathfrak{E}a) = 1$ iff $v(a) \in D_w$.
$\Sigma \vDash A$ iff for every $I = \langle D, W, R, v\rangle$, and every $w \in W$: (If $v_w(B) = 1$, for all $B \in \Sigma$,then $v_w(A) = 1$).

$$\begin{tikzcd}
& \forall xA, i \arrow[dl] \arrow[dr] & & \exists xA, i \arrow[d] \\
\neg\mathfrak{E}a, i & & A_x(a), i & \mathfrak{E}c, i \arrow[d] \\
& & & A_x(c), i
\end{tikzcd}$$

Syntax: $\delta = \frac{\varphi:\psi_1,\ldots,\psi_n}{\chi}$ where $\text{pre}(\delta) = \varphi, \text{just}(\delta) = \{\psi_1, \ldots, \psi_n\}, \text{cons}(\delta) = \chi$.
Pi $= (\delta_0, \delta_1, \ldots)$ with $\delta_0, \delta_1, \ldots \in D$,s.t. for all $i, j : \delta_i \neg \delta_j$ if $i \neq j$.

- $\text{In}(\Pi) = \text{Th}(M)$ with $M = W \cup \{\text{cons}(\delta) | \delta \in \Pi\}$
- δ applicable to S, if $\text{pre}(\delta) \in S$, and $\neg\psi_i \notin S$, for all $\psi_i \in \text{just}(\delta)$
- Π is called proces of T, iff δ_k app. to $\text{In}(\Pi[k])$ for every k s.t. $\delta_k \in \Pi$
- Π closed if you can not apply any δ_k anymore.
- $\text{Out}(\Pi) = \{\neg\Psi | \text{There is } \delta \in \Pi \text{ s.t. } \psi \in \text{just}(\delta)\}$

- Π successful if $\mathrm{In}(\Pi) \cap \mathrm{Out}(\Pi) = \varnothing$. Otherwise failed.

Set formula E, is extension to default T, fif there si some closed and successful process of T s.t. $E = \mathrm{In}(\Pi)$.
$(W, D) \mid\!\sim_s$ iff φ in all extensions of (W, d)
$(W, D) \mid\!\sim_c$ iff φ in at least one extension of (W, D).
If T has no extensions, then $(W, D) \mid\!\sim_s \varphi$ for every φ, and $(W, D) \mid\!\sim_c \varphi$ for no formula φ.